Math Camp 2025: Session 5
Comparative Statics

Jinpeng Shi

Department of Economics, George Mason University

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Introduction

Comparative-static analysis, more commonly known as *comparative statics*, compares the different equilibrium values of the endogenous variables resulting from changes in the values of the exogenous variables and parameters in the model

- e.g., the responsiveness of consumer demand to a projected excise tax, tariff, or subsidy

Comparative Statics with One Endogenous Variable

Assume a general model in which the supply and demand of a commodity are given solely by general functions

$$\text{Demand} = D(P, Y)$$
$$\text{Supply} = S(P)$$

- where $D_P < 0, D_Y > 0, S_P > 0$

The equilibrium price level P^* can be found where demand equals supply

$$D(P, Y) = S(P)$$

Thus, excess demand equals zero

$$F(P, Y) = D(P, Y) - S(P) = 0$$

Finding the Comparative-static Derivatives

Thus, from total differentials,

$$\frac{dP^*}{dY} = -\frac{F_Y}{F_P}$$

- substituting terms, if it is a normal good $D_Y > 0$

$$\frac{dP^*}{dY} = -\frac{D_Y}{D_P - S_P} = -\frac{(+)}{(-) - (+)} > 0$$

However, if it is a inferior good but not a Giffen good, where $D_Y \leq 0, D_P < 0$, then

$$\frac{dP^*}{dY} \leq 0$$

Comparative Statics with More Than One Endogenous Variable

In a model with more than one endogenous variable, comparative statics requires that there be a unique equilibrium condition for each of the endogenous variables

- *n* endogenous variables, *n* equilibrium conditions

Consider a model with two exogenous variables x_1, x_2, and two endogenous variables y_1, y_2

$$F(y_1, y_2; x_1, x_2) = 0$$
$$G(y_1, y_2; x_1, x_2) = 0$$

To find the comparative-static partial derivatives of the system with respect to x_1

$$\frac{\partial F}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial F}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} + \frac{\partial F}{\partial x_1} = 0$$
$$\frac{\partial G}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial G}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} + \frac{\partial G}{\partial x_1} = 0$$

Comparative Statics with More Than One Endogenous Variable

Express the system of linear equations in matrix notation

$$\begin{bmatrix} \frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} \\ \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} \\ \frac{\partial y_2}{\partial x_1} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F}{\partial x_1} \\ -\frac{\partial G}{\partial x_1} \end{bmatrix}$$

where,

$$JX = B$$

If both functions have continuous first and second derivatives and the Jacobian $|J|$ is nonzero, then according to Cramer's rule:

$$\frac{\partial y_i}{\partial x_1} = \frac{|J_i|}{|J|}$$

where J_i is replacing the *i*th column of Jacobian using the column vector B

Comparative Statics for Optimization Problems

Economists may also want to study the effects of changes in exogenous variables on the solution values of optimization problems

- A price-taking firm has a strictly concave production function $Q(K, L)$. Given P output price, r rental rate of capital, and w wage, its profit function is

$$\pi = PQ(K, L) - rK - wL$$

- the first-order optimization conditions are

$$\frac{\partial \pi}{\partial K} = PQ_K(\bar{K}, \bar{L}) - r = 0$$
$$\frac{\partial \pi}{\partial L} = PQ_L(\bar{K}, \bar{L}) - w = 0$$

Comparative Statics for Optimization Problems: Continued

To determine the effects of a change in the exogenous variables (r, w) on the optimal values of the endogenous variables $(\bar{K}, \bar{L})$

- Take the total derivatives of the first-order conditions with respect to either of the exogenous variables and set them in the matrix form

$$\begin{bmatrix} PQ_{KK} & PQ_{KL} \\ PQ_{KL} & PQ_{LL} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{K}}{\partial r} \\ \frac{\partial \bar{L}}{\partial r} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- $|J| \neq 0$, since it is equivalent to the Hessian of this optimization problem

- Solve comparative statics using Cramer's rule:

$$\frac{\partial \bar{K}}{\partial r} = \frac{|J_1|}{|J|} = \frac{\begin{vmatrix} 1 & PQ_{KL} \\ 0 & PQ_{LL} \end{vmatrix}}{\begin{vmatrix} PQ_{KK} & PQ_{KL} \\ PQ_{KL} & PQ_{LL} \end{vmatrix}} = \frac{PQ_{LL}}{P^2(Q_{KK}Q_{LL} - Q_{KL}^2)} < 0$$

Comparative Statics Used in Constrained Optimization

The previous procedure can also be applied to constrained optimization problems

- Assume a firm operating in perfectly competitive input and output markets wants to maximize its output $q(K, L)$ subject to a given budgetary constraint

$$rK + wL = B$$

- The Lagrangian function is

$$Q(K, L) = q(K, L) + \lambda(rK + wL - B)$$

- Now the first-order derivatives are

$$\begin{aligned}
\frac{\partial Q}{\partial K} &= q_K(\bar{K}, \bar{L}) - r\bar{\lambda} = 0 \\
\frac{\partial Q}{\partial L} &= q_L(\bar{K}, \bar{L}) - w\bar{\lambda} = 0 \\
\frac{\partial Q}{\partial \lambda} &= r\bar{K} + w\bar{L} - B = 0
\end{aligned}$$

Comparative Statics Used in Constrained Optimization: Continued

To find the effect of a change in the budget B on the optimal values of the endogenous variables, we take the total derivative of each of the three functions with respect to B

- new Jacobain is

$$\begin{bmatrix} q_{KK} & q_{KL} & -r \\ q_{KL} & q_{LL} & -w \\ r & w & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{K}}{\partial B} \\ \frac{\partial \bar{L}}{\partial B} \\ \frac{\partial \bar{\lambda}}{\partial B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Then use Cramer's rule to derive comparative statics $\frac{\partial \bar{K}}{\partial B}$

The Envelope Theorem

The envelope theorem states that the partial derivative of the indirect objective function with respect to any one of the exogenous variables, equals the partial derivative of the Lagrangian function with respect to the same exogenous variable

- Assume one wishes to maximize $z(x, y; a, b)$
- Subject to $f(x, y; a, b) = 0$
- The Lagrangian function is

$$Z(x, y, \lambda; a, b) = z(x, y; a, b) + \lambda f(x, y; a, b)$$

- when evaluated at optimal values

$$Z(\bar{x}(a, b), \bar{y}(x, y), \bar{\lambda}; a, b) = z[\bar{x}(a, b), \bar{y}(x, y); a, b] + \bar{\lambda}(a, b) f[\bar{x}(a, b), \bar{y}(x, y); a, b]$$

The Envelope Theorem: Continued

The objective function, when evaluated at the values of the optimal solution, is known as the *indirect objective function*

$$\bar{z}(a,b) \equiv z[\bar{x}(a,b), \bar{y}(x,y); a, b]$$

Taking the partial derivative on both sides with respect to b

$$\frac{\partial \bar{z}}{\partial b} = z_x \frac{\partial \bar{x}}{\partial b} + z_y \frac{\partial \bar{y}}{\partial b} + z_b$$

Recall from first-order conditions

$$z_x = -\bar{\lambda} f_x, \quad z_y = -\bar{\lambda} f_y$$

Thus,

$$\frac{\partial \bar{z}}{\partial b} = -\bar{\lambda}(f_x \frac{\partial \bar{x}}{\partial b} + f_y \frac{\partial \bar{y}}{\partial b}) + z_b$$

The Envelope Theorem: Continued

Taking the partial derivative of the constraint with respect to *b*

$$f_x \frac{\partial \bar{x}}{\partial b} + f_y \frac{\partial \bar{y}}{\partial b} + f_b = 0$$

Thus, substitute back into $\frac{\partial \bar{z}}{\partial b}$

$$\frac{\partial \bar{z}}{\partial b} = \bar{\lambda} f_b + z_b = \frac{\partial Z}{\partial b} \qquad \square$$

The derivative of the Lagrangian function with respect to a specific exogenous variable, when evaluated at the optimal values of the problem, is a reliable measure of the effect of that exogenous variable on the optimal value of the objective function