Math Camp 2025: Session 4
Differential Equations, Optimization

Jinpeng Shi

Department of Economics, George Mason University

August 21, 2025

First-order Differential Equations

A *differential equation* is an equation which expresses an explicit or implicit relationship between a function $y = f(t)$ and one or more of its derivatives or differentials

$$\frac{dy}{dt} = 5t + 9 \quad \text{or} \quad y'' + y' + 19 = 0$$

- Equations involving a single independent variable are called *ordinary differential equations*
- The *order* of a differential equation is the order of the highest derivative in the equation
- The *degree* of a differential equation is the highest power to which the derivative of highest order is raised

Differential Equations: Order and Degree

- first order, first degree:

$$\frac{dy}{dt} = 5t + 9$$

- second order, first degree:

$$\frac{d^2y}{dt^2} = 5t + 9$$

- first order, second degree:

$$(\frac{dy}{dt})^2 = 5t + 9$$

Solving Differential Equations

suppose $y''(t) = 7$

- integrate both sides

$$y'(t) = \int 7 dt = 7t + c_1$$

- integrate once more

$$y(t) = \int 7t + c_1 dt = \frac{7}{2}t^2 + c_1 t + c$$

General Formula for First-order Linear Differential Equations

For a first-order *linear* differential equation, $\frac{dy}{dt}$ and y must be of the first degree, and no product $y\frac{dy}{dt}$ may occur

$$\frac{dy}{dt} + vy = z$$

- a general solution is

$$y(t) = e^{-\int v dt}(A + \int z e^{\int v dt} dt)$$

More on General Formula

$$y(t) = e^{-\int v dt}(A + \int z e^{\int v dt} dt)$$

- A is an arbitrary constant
- $e^{-\int v dt} A$ is called the *complementary function*
- $e^{-\int v dt} \int z e^{\int v dt} dt$ is called the *particular integral*
- The particular integral y_p equals the *intertemporal equilibrium* level of $y(t)$
- the complementary function y_c represents the *deviation* from the equilibrium

For $y(t)$ to be *dynamically stable*

- y_c must approach zero as t approaches infinity

Exact Differential Equations and Partial Integration

Given a function of more than one independent variable, such as $F(y, t)$ where $M = \frac{\partial F}{\partial y}$ and $N = \frac{\partial F}{\partial t}$, the total differential is written

$$dF(y, t) = Mdy + Ndt$$

- If the differential is set equal to zero, it is called an *exact differential equation*
- For an exact differential equation, $\frac{\partial M}{\partial t}$ must equal $\frac{\partial N}{\partial y}$

Solution of an exact differential equation calls for successive integration with respect to one independent variable at a time while holding constant the other independent variable(s)

Solving Exact Nonlinear Differential Equations

Consider $(6yt + 9y^2)dy + (3y^2 + 8t)dt = 0$

- First, let $M = 6yt + 9y^2$, let $N = 3y^2 + 8t$

  $$\frac{\partial M}{\partial t} = 6y \quad \frac{\partial N}{\partial y} = 6y$$

  Thus, exact differential equation

- Since $M = \frac{\partial F}{\partial y}$ is a partial derivative, integrate M partially with respect to y by treating t as a constant, and add a new function $Z(t)$ for any additive terms of t which would have been eliminated by the original differentiation with respect to y

  $$F(y,t) = \int M dy + Z(t) = 3y^2 t + 3y^3 + Z(t)$$

Solving Exact Nonlinear Differential Equations: Continued

- Differentiate the previous equation with respect to t to find $\frac{\partial F}{\partial t}$

$$\frac{\partial F}{\partial t} = 3y^2 + Z'(t)$$

- Note this partial derivative must equal to N

$$3y^2 + Z'(t) = 3y^2 + 8t$$

- Thus, integrate $Z(t)$

$$Z(t) = \int 8t = 4t^2 + c$$

Therefore,

$$F(y,t) = 3y^2 + t + 3y^3 + 4t^2 + c$$

Separation of Variables

If, however, the equation can be written in the form of separated variables such that $R(y)dy + S(t)dt = 0$, where R and S, respectively, are functions of y and t alone,

- Consider $\frac{dy}{dt} = y^2 t$
- First, separate variables

$$\frac{dy}{y^2} = tdt$$

- Then integrate both sides,

$$\int \frac{dy}{y^2} = \int tdt$$

The equation can be solved simply by ordinary integration

Optimization of Functions

Optimization is the process of finding the relative maximum or minimum of a function

- Given the usual differentiable function, take the first derivative, set it equal to zero, and solve for the critical point(s). All such points are candidates for a possible relative maximum or minimum.

- This step represents a necessary condition known as the *first-order condition*

- Take the second derivative, evaluate it at the critical point(s), and check the sign(s). If at a critical point a

$$\begin{aligned} f''(a) &< 0, && \text{concave at } a\text{, relative maximum} \\ f''(a) &> 0, && \text{convex at } a\text{, relative minimum} \\ f''(a) &= 0, && \text{inconclusive} \end{aligned}$$

Optimization of Multivariable Functions

Similarly, for a multivariable function such as$z = f(x, y)$ to be at a relative minimum or maximum

- The first-order partial derivatives must equal zero simultaneously

$$f_x = 0, \quad f_y = 0$$

- The second-order direct partial derivatives, when evaluated at the critical point (a, b), must both be negative for a relative maximum and positive for a relative minimum

$$f_{xx} < 0, \quad f_{yy} < 0 \quad \text{or} \quad f_{xx} > 0, \quad f_{yy} > 0$$

- The product of the second-order direct partial derivatives evaluated at the critical point must exceed the product of the cross partial derivatives also evaluated at the critical point

$$f_{xx} \cdot f_{yy} > f_{xy}^2$$

Constrained Optimization with Lagrange Multipliers

Differential calculus is also used to maximize or minimize a function $f(x, y)$ subject to constraint $g(x, y) = k$

- setting the constraint equal to zero

$$g(x, y) - k = 0$$

- multiplying it by λ (the *Lagrange multiplier*) and adding the product to the original function

$$F(x, y) = f(x, y) + \lambda[g(x, y) - k]$$

- $F(x, y)$ is the *Lagrangian function*