Math Camp 2025: Session 3
Calculus

Jinpeng Shi

Department of Economics, George Mason University

August 20, 2025

Functions of Several Variables and Partial Derivatives

$z = f(x, y)$ is defined as a *function of two independent variables*

- if there exists one and only one value of z in the range of f for each ordered pair of real numbers (x, y) in the domain of f

The *partial derivative* of z with respect to x

- measures the instantaneous rate of change of z with respect to x while y is held constant
- can be written as $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_x, z_x$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Rules of Partial Differentiation

Treat the other independent variables as constants

- Product rule: $z = g(x, y) \cdot h(x, y)$

$$\frac{\partial z}{\partial x} = g(x, y) \cdot \frac{\partial h}{\partial x} + h(x, y) \cdot \frac{\partial g}{\partial x}$$

- Quotient rule: $z = \frac{g(x,y)}{h(x,y)}, h(x, y) \neq 0$

$$\frac{\partial z}{\partial x} = \frac{h(x, y) \cdot \frac{\partial g}{\partial x} - g(x, y) \cdot \frac{\partial h}{\partial x}}{[h(x, y)]^2}$$

- Generalized power function rule: $z = [g(x, y)]^n$

$$\frac{\partial z}{\partial x} = n \cdot [g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial x}$$

Second-order Partial Derivatives

The *second-order (direct) partial derivative* signifies that the function has been differentiated partially with respect to one of the independent variables twice while the other independent variable has been held constant

$$f_{xx} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) = \frac{\partial^2 z}{\partial x^2}$$

The cross (or mixed) partial derivatives indicate that first the primitive function has been partially differentiated with respect to one independent variable and then that partial derivative has in turn been partially differentiated with respect to the other independent

$$f_{xy} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x}) = \frac{\partial^2 z}{\partial y \partial x}$$

Young's Theorem and Differentials

If both cross partial derivatives are continuous, they will be identical

$$f_{xy} = f_{yx}$$

Differential of y, dy, measures the change in y resulting from a small change in x, written dx

- The derivative $\frac{dy}{dx}$ may also be treated as a ratio of differentials

$$\frac{dy}{dx} = 4x + 5, \quad \text{rate of change}$$
$$dy = (4x + 5)dx, \quad \text{change in} \quad y$$

Total and Partial Differentials

The *total differential* measures the change in the dependent variable brought about by a small change in each of the independent variables

$$dz = z_x dx + z_y dy$$

A *partial differential* measures the change in the dependent variable of a multivariate function resulting from a small change in one of the independent variables and assumes the other independent variables are constant

$$dz = z_x dx, \quad dy = 0$$

Total Derivatives

The *total derivative* measures

- the direct effect of x on z: $\frac{\partial z}{\partial x}$
- the indirect effect of x on z through y: $\frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$

Thus,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

Implicit Function Rules

Functions of the form $f(x, y) = 0$ do not express y explicitly in terms of x and are called *implicit functions*

- If an implicit function $f(x, y) = 0$ exists and $f_y \neq 0$ at the point around which the implicit function is defined, the total differential is

$$f_x dx + f_y dy = 0$$

- The total derivative is

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Inverse Function Rules

An inverse function $x = f^{-1}(y)$ exists if each value of y yields one and only one value of x

- the derivative of the inverse function is the reciprocal of the derivative of the original function

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \quad \text{Provided} \quad \frac{dy}{dx} \neq 0$$

Integration

Reversing the process of differentiation and finding the original function from the derivative is called *integration*

- The original function $F(x)$ is called the *integral* of $F'(x)$
- Let $f(x) = F'(x)$
$$\int f(x)dx = F(x) + c$$
- The left-hand side: the indefinite integral of f of x with respect to x
- $f(x)$ is the integrand
- c is the constant of integration
- In many problems an *initial condition* ($y = y_0$ when $x = 0$) or a *boundary condition* ($y = y_0$ when$x = x_0$) is given which uniquely determines the constant of integration

Rules of Integration

The derivative of the integral must equal the integrand

- The integral of a constant k

$$\int k dx = kx + c$$

- The integral of a power function x^n, where $n \neq -1$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$

- The integral of x^{-1}, $x > 0$

$$\int x^{-1} dx = \ln x + c$$

Rules of Integration: Continued

- The integral of an exponential function a^{kx}

$$\int a^{kx} dx = \frac{a^{kx}}{k \ln a} + c$$

- The integral of a constant times a function equals the constant times the integral of the function

$$\int kf(x) dx = k \int f(x) dx$$

- The integral of the sum or difference of two or more functions equals the sum or difference of their integrals

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

Integration by substitution

By expressing the integrand $f(x)$ as a function of u and its derivative $\frac{du}{dx}$ and integrating with respect to x

$$\int f(x)dx = \int u\frac{du}{dx}dx = \int udu = F(u) + c$$

- Consider an indefinite integral: $\int 12x^2(x^3 + 2)dx$
- Let $u = x^3 + 2$, thus
$$\frac{du}{dx} = 3x^2$$
- Substitute u for $x^3 + 2$, $\frac{du}{3x^2}$ for dx

$$\int 12x^2(x^3 + 2)dx = \int 12x^2 \cdot u \cdot \frac{du}{3x^2} = 4\int udu = 2u^2 + c = 2(x^3 + 2)^2 + c$$

Integration by Parts

Recall product rule for differentiation

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

- Taking the integral of the derivative gives

$$f(x)g(x) = \int f(x)g'(x)dx + \int f'(x)g(x)dx$$

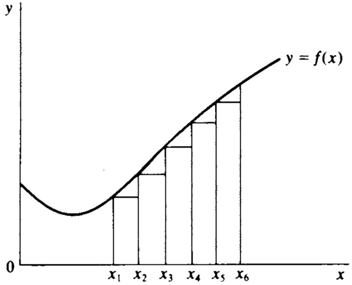
- Thus,

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Area under a Curve

the sum of the areas of the rectangles $\sum_{i=1}^{n} f(x_i)\Delta x_i$, called a *Riemann sum*, will approximate, but underestimate, the actual area under the curve

- The smaller the subintervals, the closer to the actual area under the curve



(b)

Definite Integral

Thus, if the number of subintervals is increased so that $n \to \infty$, each subinterval becomes infinitesimal ($\Delta x_i = dx_i = dx$) and the area A under the curve can be expressed as

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

- more succinctly as the *definite integral*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

- the left-hand side: the integral from a to b of f of xdx
- a is called the lower limit of integration and b the upper limit of integration

The Fundamental Theorem of Calculus

Unlike the indefinite integral which is a set of functions containing all the integrals of $f(x)$

- The *fundamental theorem of calculus* states that the numerical value of the definite integral of a continuous function $f(x)$ over the interval from a to b is given by the indefinite integral $F(x) + c$ evaluated at the upper limit of integration b, minus the same indefinite integral $F(x) + c$ evaluated at the lower limit of integration a
- the constant of integration is eliminated in subtraction

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Properties of Definite Integral

- Reversing the order of the limits changes the sign of the definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero

$$\int_a^a f(x)dx = F(a) - F(a) = 0$$

- The definite integral can be expressed as the sum of component subintegrals

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad a \leq c \leq b$$

Improper Integrals

A definite integral with infinity for either an upper or lower limit of integration is called an *improper integral*

$$\int_a^{\infty} f(x)dx$$

They can be defined as the limits of other integrals

$$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^{b} f(x)dx$$

- If the limit exists, the improper integral is said to *converge*
- The integral has a definite value

L'Hospital's Rule

If the limit of a function $f(x) = \frac{g(x)}{h(x)}$ as $x \rightarrow a$ cannot be evaluated

- when both numerator and denominator approach zero, giving rise to the indeterminate form $\frac{0}{0}$
- when both numerator and denominator approach infinity, giving rise to the indeterminate form $\frac{\infty}{\infty}$

L'Hospital's rule states:

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{g'(x)}{h'(x)}$$

Definite Integral and Probability

The probability P that an event will occur can be measured by the corresponding area under a probability density function $f(x)$ such that

- Probability cannot be negative: $f(x) \geq 0$
- The probability of the event occurring over the entire range of x is 1: $\int_{-\infty}^{\infty} f(X)dx = 1$
- The probability of the value of x falling within the interval $[a, b]$ is the value of the definite integral from a to b: $P(a < x < b) = \int_a^b f(x)dx$