

# Math Camp 2025: Session 2

## Linear Algebra

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# Matrix

A *matrix* is a rectangular array of numbers, parameters, or variables

- The numbers (parameters, or variables) are referred to as *elements* of the matrix
- The numbers in a horizontal line are called *rows*; the numbers in a vertical line are called *columns*
- The number of rows  $r$  and columns  $c$  defines the dimensions of the matrix ( $r \times c$ )

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}_{2 \times 3}$$

## Matrix: Continued

- In a *square matrix*, the number of rows equals the number of columns (that is,  $r = c$ )

$$B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}_{2 \times 2}$$

- If the matrix is composed of a single column, it is a column vector; similarly for row vector

$$C = \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2 \times 1}$$

# Transpose

A matrix which converts the rows of  $A$  to columns and the columns of  $A$  to rows is called the *transpose* of  $A$

- designated by  $A'$  or  $A^T$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}_{2 \times 3} \quad A^T = \begin{bmatrix} 1 & a \\ 2 & b \\ 3 & c \end{bmatrix}_{3 \times 2}$$

## Addition and Subtraction

Addition (and subtraction) of two matrices  $A + B$  (or  $A - B$ ) requires that the matrices be of equal dimensions

- Each element of one matrix is then added to (subtracted from) the corresponding element of the other matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 2 & 2 & 1 \\ c & b & 2a \end{bmatrix}_{2 \times 3}$$

Thus,

$$A + B = \begin{bmatrix} 1 + 2 & 2 + 2 & 3 + 1 \\ a + c & b + b & c + 2a \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 3 & 4 & 4 \\ a + c & 2b & c + 2a \end{bmatrix}_{2 \times 3}$$

# Scalar Multiplication

A simple number such as 2 is called a *scalar*

- Multiplication of a matrix by a number or scalar involves multiplication of every element of the matrix by the number

Suppose  $k = 8$ , thus

$$kA = \begin{bmatrix} 8 & 16 & 24 \\ 8a & 8b & 8c \end{bmatrix}_{2 \times 3}$$

# Vector Multiplication

Multiplication of a row vector  $A$  by a column vector  $B$  requires as a precondition that each vector have precisely the same number of elements.

$$A = [1 \quad 2 \quad 3]_{1 \times 3} \quad B = \begin{bmatrix} 2 \\ c \\ 5 \end{bmatrix}_{3 \times 1}$$

Thus,

$$AB = 1 \times 2 + 2 \times c + 3 \times 5 = 17 + 2c$$

# Matrix Multiplication

Multiplication of two matrices requires that the matrices be *conformable*

- the number of columns in the *lead matrix*, equal the number of rows in the *lag matrix*
- The row-column products, called *inner products* or *dot products*, are then used as elements in the formation of the product matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 2 & 2 \\ c & b \\ 3 & a \end{bmatrix}_{3 \times 2}$$

$c_A = r_B$ , thus  $A$  and  $B$  are conformable

# Commutative, Associative, and Distributive Laws

Matrix addition is commutative and associative:

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$

Matrix multiplication is not commutative, but associative and distributive

- $AB \neq BA$
- $(XY)Z = X(YZ)$
- $A(B + C) = AB + AC$

However, scalar multiplication is commutative

- $kA = Ak$

## Identity and Symmetric Matrix

An identity matrix  $I$  is a square matrix which has 1 for every element on the principal diagonal from left to right and 0 everywhere else

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any matrix for which  $A = A^T$  is a *symmetric* matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

## Determinants: second-order

The determinant  $|A|$  of a  $2 \times 2$  matrix, called a second-order determinant

- is derived by taking the product of the two elements on the principal diagonal and subtracting from it the product of the two elements off the principal diagonal

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$|A| = 1 \times 1 - 2 \times 2 = -3$$

# Higher-order Determinants and Laplace Expansion

The determinant of a  $3 \times 3$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Take the first element of the first row,  $a_{11}$ , and mentally delete the row and column in which it appears, the remaining matrix is called a *minor* ( $M_{11}$ )
- Then multiply  $a_{11}$  by the determinant of the minor  $a_{11}|M_{11}|$
- In general, there is a prescribed sign before the determinants of the minor  $(-1)^{i+j}$

Thus,

$$|A| = \sum_{j=1}^n (-1)^{i+j} a_{ij} |M_{ij}|$$

# Singularity and Rank

The determinant of a matrix is equal to zero,  $|A| = 0$

- A singular matrix is one in which there exists linear dependence between at least two rows or columns.

The rank  $r$  of a matrix is defined as the maximum number of linearly independent rows or columns in the matrix

- Assuming a square matrix of order  $n$

$r(A) = n$ ,  $A$  is nonsingular and no linear dependence

$r(A) < n$ ,  $A$  is singular and there is linear dependence

# Inverse

An inverse matrix  $A^{-1}$ , which can be found only for a square, nonsingular matrix  $A$

$$AA^{-1} = I = A^{-1}A$$

- a *cofactor* is a minor with prescribed sign:  $|C_{ij}| = (-1)^{i+j}|M_{ij}|$
- A *cofactor matrix* is a matrix in which every element  $a_{ij}$  is replaced with its cofactor  $|C_{ij}|$
- An *adjoint matrix* is the transpose of a cofactor matrix  $A_{adj} = C'$

Thus, the inverse is

$$A^{-1} = \frac{1}{|A|} \cdot A_{adj}$$

# Matrix Expression of a System of Linear Equations

Matrix algebra permits the concise expression of a system of linear equations

$$7a + 3b = 45$$

$$4a + 5b = 29$$

can be expressed in a matrix form  $AX = B$ , where

$$A = \begin{bmatrix} 7 & 3 \\ 4 & 5 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \end{bmatrix} \quad B = \begin{bmatrix} 45 \\ 29 \end{bmatrix}$$

- $A$  is the *coefficient matrix*
- $X$  is the *solution vector*
- $B$  is the *vector of constant terms*

# Solving Linear Equations

An inverse matrix can be used to solve matrix equations

$$A_{n \times n} X_{n \times 1} = B_{n \times 1}$$

- if  $A^{-1}$  exists, then  $A_{n \times n}^{-1} A_{n \times n} X_{n \times 1} = A_{n \times n}^{-1} B_{n \times 1}$
- since  $AA^{-1} = I$ , thus  $I_{n \times n} X_{n \times 1} = A_{n \times n}^{-1} B_{n \times 1}$

Therefore,

$$X_{n \times 1} = A_{n \times n}^{-1} B_{n \times 1}$$

## Cramer's Rule

Cramer's rule states  $x_i = \frac{|A_i|}{|A|}$

- $|A_i|$  is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of  $x_i$  with the column vector of constants

Recall,

$$A = \begin{bmatrix} 7 & 3 \\ 4 & 5 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \end{bmatrix} \quad B = \begin{bmatrix} 45 \\ 29 \end{bmatrix}$$

To solve for  $a$ , replace column 1, the coefficients of  $x_1$ , with the vector of constants  $B$ , forming a new matrix  $A_1$

$$A_1 = \begin{bmatrix} 45 & 3 \\ 29 & 5 \end{bmatrix}$$

$$\text{Thus, } a = \frac{|45 \times 5 - 3 \times 29|}{|7 \times 5 - 3 \times 4|} = \frac{138}{23}$$