Math Camp 2025: Session 1
Review of Basic Algebra, Derivative

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Exponents

Given n a positive integer, x^n signifies that x is multiplied by itself n times.

- any nonzero number or variable raise to the zero power equal to 1

$$\text{e.g.} \quad \frac{x^3}{x^3} = x^{3-3} = x^0 = 1$$

- 0^0 is undefined
- in multiplication, exponents of the same variable are added; in division, exponents are subtracted

$$x^2(x^3) = x^{2+3} = x^5 \neq x^6$$

- when raised to power, exponents are multiplied

$$(x^4)^2 = x^{4 \cdot 2} = x^8 = x^4 \cdot x^4$$

Polynomials

Given an expression such as $5x^3$

- x is called a variable because it can assume any number of different values
- 5 is called coefficient

Expressions consisting simply of a real number or of a coefficient times one or more variables raised to the power of a positive integer called *monomials*

- monomials can be added or subtracted to form **polynomials**
- each monomial comprising a polynomial is called *term*
- terms have same variables and exponents are called *like terms*

Rules of Polynomials

- like terms in polynomials can be added or subtracted by adding their coefficients

$$4x^5 + 9x^5 = 13x^5$$

- like and unlike terms can be multiplied or divided by multiplying or dividing both the coefficients and variables

$$5x \cdot 13y^2 = 65xy^2$$

- in multiplying two polynomials, each term in the first polynomial must be multiplied by each term in the second and their products added

$$\begin{aligned}(6x + 7y)(4x + 9y) &= 24x^2 + 54xy + 28xy + 63y^2 \\ &= 24x^2 + 82xy + 63y^2\end{aligned}$$

Linear Equations

A mathematical statement setting two algebraic expressions equal to each other is called an *equation*

- an equation in which all variables are raised to the first power is known as a *linear equation*

$$\frac{x}{4} - 3 = \frac{x}{5} + 1$$

A linear equation can be solved by moving the unknown variable to the left-hand side of the equal sign and all the other terms to the right-hand side

$$\frac{x}{4} - \frac{x}{5} = 1 + 3$$

Simplify both sides of the equation until the unknown variable is by itself on the left and the solution is on the right

$$x = 80$$

Quadratic Equations

A quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$, can be solved using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- the quadratic function can also be solved by factoring

$$5x^2 - 55x + 140 = 0$$
$$(5x - 20)(x - 7) = 0$$
$$\text{Thus,} \quad x_1 = 4, x_2 = 7$$

Simultaneous Equations

To solve a system of two or more equations simultaneously

- the equations must be *consistent* (noncontradictory)
- they must be *independent* (not multiples of each other)
- there must be as many consistent and independent equations as variables

A system of simultaneous linear equations can be solved by either the substitution or elimination method

Substitution Method

$$\begin{aligned} 8b - 3m &= 7 \\ -b + 7m &= 19 \end{aligned}$$

- solve one of the equations for one variable in terms of the other

$$b = 7m - 19$$

- substitute the value of that term in the other equation

$$\begin{aligned} 8(7m - 19) - 3m &= 7 \\ m &= 3 \end{aligned}$$

- then substitute back m

$$b = 7 \cdot 3 - 19 = 2$$

Elimination Method

- multiply the second equation by coefficient of 8

$$8b - 3m = 7$$
$$-8b + 56m = 152$$

- add two equations together to eliminate b

$$53m = 159$$

- solve m and b

$$m = 3$$
$$b = 2$$

Functions

A function f is a *rule* which assigns to each value of a variable (x) one and only one value [$f(x)$]

- x: an *argument* of a function is a value provided to obtain the function's result
- [$f(x)$] referred to as the value of the function at x
- The *domain* of a function refers to the set of all possible values of x
- the *range* is the set of all possible values for $f(x)$

Linear function:

$$f(x) = mx + b$$

Graphs, Slopes and Intercepts

In graphing a function such as $y = f(x)$

- x is placed on the horizontal axis and is known as the *independent variable*
- y is placed on the vertical axis and is called the *dependent variable*
- The graph of a linear function is a straight line
- The slope indicates the steepness and direction of a line; the greater the absolute value of the slope, the steeper the line
- The y intercept is the point where the graph crosses the y axis; it occurs when $x = 0$
- The x intercept is the point where the line intersects the x axis; it occurs when $y = 0$

Limits

If the functional values $f(x)$ of a function f draw closer to one and only one finite real number L for all values of x as x draws closer to a from both sides, but does not equal a, L is defined as the limit of $f(x)$ as x approaches a

$$\lim_{x \to a} f(x) = L$$

- $\lim_{x \to a} k = k$: k is a constant
- $\lim_{x \to a} x^n = a^n$: n is a positive integer
- $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$:$k$ is a constant
- $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$: $n > 0$

Continuity

A *continuous* function is one which has no breaks in its curve. A function f is continuous at $x = a$ if:

- $f(x)$ is defined
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

All polynomial functions are continuous, as are all rational functions, except where undefined (denominators equal zero).

The Derivative

Given a function $y = f(x)$, the *derivative* of the function f at x, written $f'(x)$ is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{if the limit exists}$$

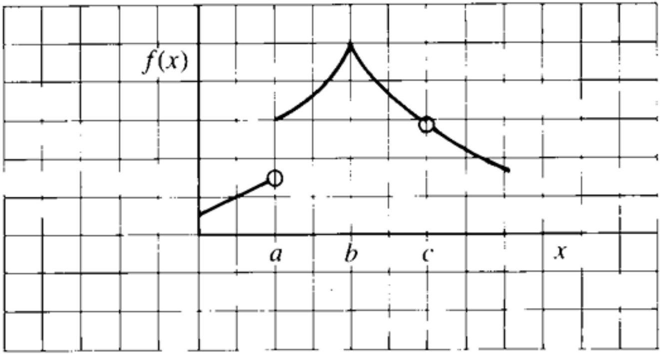
A function is differentiable at a point if the derivative exists (may be taken) at that point. To be differentiable at a point, a function must

- be continuous at that point
- have a unique tangent at that point

The same logic for higher-order derivatives

Differentiability and Continuity

Continuity alone, however, does not ensure (is not a sufficient condition for) differentiability



Rules of Differentiation

- Constant Function Rule: $f(x) = k$

$$f'(x) = 0$$

- Linear Function Rule: $f(x) = mx + b$

$$f'(x) = m$$

- Power Function Rule: $f(x) = kx^n$

$$f'(x) = k \cdot n \cdot x^{n-1}$$

Rules of Differentiation: Continued

- Rules for Sums and Differences: $f(x) = g(x) \pm h(x)$

$$f'(x) = g'(x) \pm h'(x)$$

- Product Rule: $f(x) = g(x) \cdot h(x)$

$$f'(x) = g(x) \cdot h'(x) + g'(x) \cdot h(x)$$

- Quotient Rule: $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Rules of Differentiation: Continued

- Generalized Power Function Rule: $f(x) = [g(x)]^n$

$$f'(x) = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

- Chain Rule: $y = f(u), u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Implicit Differentiation

Given $3x^4 - 7y^5 - 86 = 0$

- Differentiate both sides of the equation with respect to x while treating y as a function of x

$$\frac{d}{dx}(3x^4 - 7y^5 - 86) = \frac{d}{dx}(0)$$

$$12x^3 - 35y^4 \cdot \frac{d}{dx}(y) = 0$$

- simply solve algebraically

$$\frac{dy}{dx} = \frac{12x^3}{35y^4}$$

Increasing and Decreasing Functions

A function $f(x)$ is said to be increasing (decreasing) at $x = a$ if in the immediate vicinity of the point $[a, f(a)]$ the graph of the function rises (falls) as it moves from left to right.

- since the first derivative measures the rate of change and slope of a function

$$
\begin{aligned}
f'(a) > 0: &\quad \text{increasing function at} \quad x = a \\
f'(a) < 0: &\quad \text{decreasing function at} \quad x = a
\end{aligned}
$$

Concavity and Convexity

A function $f(x)$ is concave (convex) at $x = a$ if in some small region close to the point $[a, f(a)]$ the graph of the function lies completely below (above) its tangent line.

- we use second derivative to determine convexity of functions

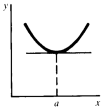
$$\begin{aligned} f''(a) > 0 : &\quad f(x) \quad \text{is convex at} \quad x = a \\ f''(a) < 0 : &\quad f(x) \quad \text{is concave at} \quad x = a \end{aligned}$$

Relative Extrema

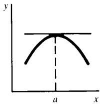
A *relative extremum* is a point at which a function is at a relative maximum or minimum

- the function must be neither increasing nor decreasing at a
- the first derivative of the function at a must equal zero or be undefined
- A point in the domain of a function where the derivative equals zero or is undefined is called a *critical point*

$$f'(a) = 0 \quad f''(a) > 0: \quad \text{relative minimum at} \quad x = a$$
$$f'(a) = 0 \quad f''(a) < 0: \quad \text{relative maximum at} \quad x = a$$



$f'(a) = 0$
$f''(a) > 0$
Relative Minimum at $x = a$



$f'(a) = 0$
$f''(a) < 0$
Relative Maximum at $x = a$